ON REPRESENTATION OF AN INTEGER AS THE SUM OF THREE SQUARES AND TERNARY QUADRATIC FORMS WITH THE DISCRIMINANTS p^2, $16p^2$

ALEXANDER BERKOVICH AND WILL JAGY

ABSTRACT. Let $s(n)$ be the number of representations of n as the sum of three squares. We prove a remarkable new identity for $s(p^2n)-ps(n)$ with p being an odd prime. This identity makes nontrivial use of ternary quadratic forms with discriminants p^2, $16p^2$. These forms are related by Watson's transformations. To prove this identity we employ the Siegel–Weil and the Smith–Minkowski product formulas.

1. INTRODUCTION

Let $(a, b, c, d, e, f)(n)$ denote the number of integral representations of n by the positive ternary quadratic form $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$. We will take $(a, b, c, d, e, f)(n) = 0$, whenever $n \notin \mathbf{N}$. The discriminant Δ of a ternary form $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ is defined as

$$\Delta = \frac{1}{2}\det\begin{bmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{bmatrix} = 4abc + def - ad^2 - be^2 - cf^2.$$

We say that two ternary quadratic forms $\tilde{f}(x, y, z)$ and $\tilde{g}(x, y, z)$ with the discriminant Δ are in the same genus if they are equivalent over $\mathbf{Q}$ via a matrix in $GL(3, \mathbf{Q})$ whose entries have denominators prime to 2Δ. We add that this is the case if and only if these forms are equivalent over the real numbers and over the p-adic integers $\mathbf{Z}_p$ for all primes p [6], [9], [13].

It is well known that all ternary forms with discriminant 4 are equivalent to $x^2 + y^2 + z^2$ [8], [10]. Let p be an odd prime. Lehman derived elegant counting formulas for ternary genera in [11]. Using his results, it is straightforward to check that all ternary forms with the discriminant p^2 belong to the same genus, say $TG_{1,p}$. There are twelve genera of ternary forms with the discriminant $16p^2$. However, if one imposes additional constraints on the forms with $\Delta = 16p^2$, namely

$$(a, b, c, d, e, f)(n) = 0, \text{ when } n \equiv 1, 2 \pmod 4,$$

$$d \equiv e \equiv f \equiv 0 \pmod 2,$$

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then we will show in Section 6 that all these ternaries belong to the same genus, say $TG_{2,p}$. In Section 8 we will show how to relate $TG_{1,p}$ and $TG_{2,p}$ and Watson's transformation.

Let $s(n)$ denote the number of representations of n by ternary form $x^2+y^2+z^2$, so

$$s(n) = (1,1,1,0,0,0)(n).$$

In [3] the first author utilized q-series techniques to prove the following two theorems:

Theorem 1.1.

$$s(9n) - 3s(n) = 2(1,1,3,0,0,1)(n) - 4(4,3,4,0,4,0)(n). \tag{1.1}$$

Theorem 1.2.

$$s(25n) - 5s(n) = 4(2,2,2,-1,1,1)(n) - 8(7,8,8,-4,8,8)(n). \tag{1.2}$$

Our main object here is to prove the following

Theorem 1.3. *Let p be an odd prime, then*

$$s(p^2n) - ps(n) = 48 \sum_{\tilde{f} \in TG_{1,p}} \frac{R_{\tilde{f}}(n)}{|Aut(\tilde{f})|} - 96 \sum_{\tilde{f} \in TG_{2,p}} \frac{R_{\tilde{f}}(n)}{|Aut(\tilde{f})|}, \tag{1.3}$$

where $|Aut(\tilde{f})|$ denotes the number of integral automorphs of a ternary form $\tilde{f}$, $R_{\tilde{f}}(n)$ denotes the number of representations of n by $\tilde{f}$, and a sum over forms in a genus should be understood to be the finite sum resulting from taking a single representative from each equivalence class of forms.

This theorem was first stated in [3]. We remark that, somewhat similar in flavor, the so-called S-genus identities were recently discussed in [4], [5]. In what follows we will require the following

Theorem 1.4.

$$s(n) = \frac{16}{\pi}\sqrt{n}\psi(n)L(1,\chi(n))P(n), \tag{1.4}$$

where for $n = 4^a k$, $4 \nmid k$ one has

$$\psi(n) = \begin{cases} 0 & \text{if } k \equiv 7 \pmod 8, \\ 2^{-a} & \text{if } k \equiv 3 \pmod 8, \\ 3\cdot 2^{-a-1} & \text{if } k \equiv 1,2 \pmod 4; \end{cases} \tag{1.5}$$

$L(1,\chi(n)) = \sum_{m=1}^{\infty}(-4n|m)m^{-1}$ *with* $\chi(n) = (-4n|\bullet)$*, the Kronecker symbol and*

$$P(n) = \prod_{(p'^2)^b || n} \left(1 + \frac{1}{p'} + \frac{1}{p'^2} + \cdots + \frac{1}{p'^{b-1}} + \frac{1}{p'^b\big(1 - \big(-np'^{-2b}|p'\big)p'^{-1}\big)}\right), \tag{1.6}$$

with the product over all odd primes p' such that $p'^2 \mid n$.

Proofs of this Theorem may be found in [1] and [2]. We observe that $L(1,\chi(n))$ can be written as the infinite product

$$L(1,\chi(n)) = \frac{\pi^2}{8}\prod_{p'}\left(1 + \big(-n|p'\big)\frac{1}{p'}\right), \tag{1.7}$$

where p' runs through all odd primes.
Before we move on we comment that for squarefree $n \equiv 3 \pmod 8$, $n \geq 11$

$$L(1, \chi(n)) = \frac{3}{2} \frac{\pi}{\sqrt{n}} h(n),$$

where $h(n)$ is the class number of the quadratic field $Q(\sqrt{-n})$.

2. The Siegel–Weil formula for ternary quadratic forms

Let T be a genus of positive ternary forms with the discriminant Δ. Then the Siegel–Weil formula [12] implies that

$$\sum_{\tilde{f} \in T} \frac{R_{\tilde{f}}(n)}{|\mathrm{Aut}(\tilde{f})|} = 4\pi M(T) \sqrt{\frac{n}{\Delta}} \prod_{p'} d_{T,p'}(n), \tag{2.1}$$

where $|\mathrm{Aut}(\tilde{f})|$ denotes the number of integral automorphs of a ternary form $\tilde{f} = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$, while $R_{\tilde{f}}(n)$ denotes the number of representations of n by $\tilde{f}$. The sum on the left is over forms in a genus. Again, this sum (here and everywhere) should be interpreted as the finite sum resulting from taking a single representative from each equivalence class of forms. The product on the right is over all primes, the mass of the genus is defined by

$$M(T) := \sum_{\tilde{f} \in T} \frac{1}{|\mathrm{Aut}(\tilde{f})|},$$

and $d_{T,p'}(n)$ denotes the p'-adic (local) representation density, defined by

$$d_{T,p'}(n) := \frac{1}{p'^{2t}} |\{(x, y, z) \in Z^3 : ax^2 + by^2 + cz^2 + dyz + ezx + fxy \equiv n \pmod{p'^t}\}|,$$

for sufficiently large t. We comment that $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ can be chosen to be any form $\in T$. In [12] Siegel proved that when $\gcd(2\Delta, p') = 1$

$$d_{T,p'}(n) = \begin{cases} \left(\frac{1}{p'} + 1\right) + \frac{1}{p'^{k+1}}((-m|p') - 1) & \text{if } n = mp'^{2k},\ p' \nmid m, \\ \left(\frac{1}{p'} + 1\right)\left(1 - \frac{1}{p'^{k+1}}\right) & \text{if } n = mp'^{2k+1},\ p' \nmid m. \end{cases} \tag{2.2}$$

It is not hard to check that (1.4) follows easily from (2.1) and (2.2), provided one recognizes that

$$\psi(n) = d_{x^2+y^2+z^2,2}(n), \tag{2.3}$$

where $\psi(n)$ is defined in (1.5). It is easy to check that

$$d_{x^2+y^2+z^2,2}(4n) = \frac{1}{2} d_{x^2+y^2+z^2,2}(n),$$

and that

$$d_{x^2+y^2+z^2,2}(n) = 0, \quad \text{if } n \equiv 7 \pmod 8. \tag{2.4}$$

It remains to verify that

$$d_{x^2+y^2+z^2,2}(n) = \begin{cases} 1 & \text{if } n \equiv 3 \pmod 8, \\ \frac{3}{2} & \text{if } n \equiv 1, 2 \pmod 4. \end{cases} \tag{2.5}$$

This can be easily accomplished with the help of the following

Lemma 2.1. *The number of roots of*

$$x^2 \equiv c \pmod{2^t}, \quad 3 \le t, \quad c \equiv 1 \pmod 2$$

is four or zero, according as $c \equiv 1 \pmod 8$ *or* $c \not\equiv 1 \pmod 8$.

Proof of this Lemma may be found in [8]. Next, we observe that

$$|\{(x,y,z) \in Z^3 : 0 \le x,y,z < 2^t, x^2+y^2+z^2 \equiv n \pmod{2^t}\}| =$$
$$4|\{(y,z) \in Z^2 : 0 < y,z < 2^t, yz \equiv 1 \pmod 2\}| = 4 \cdot 2^{t-1} \cdot 2^{t-1},$$

when $n \equiv 3 \pmod 8$. Hence,

$$d_{x^2+y^2+z^2,2}(n) = \frac{4}{2^{2t}} 2^{t-1} 2^{t-1} = 1,$$

when $n \equiv 3 \pmod 8$. Analogously, when $n \equiv 1 \pmod 8$, $3 \le t$ we find that

$$\begin{aligned}
&|\{(x,y,z) \in Z^3 : 0 \le x,y,z < 2^t, x^2+y^2+z^2 \equiv n \pmod{2^t}\}| \\
=&3|\{(x,y,z) \in Z^3 : 0 \le x,y,z < 2^t, x \equiv 1 \pmod 2, x^2+y^2+z^2 \equiv n \pmod{2^t}\}| \\
=&3 \cdot 4|\{(y,z) \in Z^2 : 0 \le y,z < 2^t, yz \equiv 0 \pmod 2, y \equiv z \pmod 4\}| \\
=&3 \cdot 4 \cdot \frac{1}{2} \cdot 2^{t-1} \cdot 2^{t-1}.
\end{aligned}$$

And so

$$d_{x^2+y^2+z^2,2}(n) = \frac{1}{2^{2t}} \cdot 3 \cdot 4 \cdot \frac{1}{2} \cdot 2^{t-1} \cdot 2^{t-1} = \frac{3}{2} \quad \text{when } n \equiv 1 \pmod 8,$$

as desired. Other cases in (2.5) can be handled in a very similar manner.
We can now rewrite (1.4) as

$$s(n) = 2\pi\sqrt{n} \prod_{p'} d_{x^2+y^2+z^2,p'}(n).$$

Consequently,

$$s(p^2 n) - ps(n) = 2\pi\sqrt{n}\psi(n)\Gamma_p(n) \prod_{\gcd(p',2p)=1} d_{x^2+y^2+z^2,p'}(n), \tag{2.6}$$

where

$$\Gamma_p(n) := p(d_{x^2+y^2+z^2,p}(p^2 n) - d_{x^2+y^2+z^2,p}(n)).$$

From (2.2) we have at once

$$\Gamma_p(n) = \begin{cases} \frac{p-1}{p^{1+k}}(1 - (-m|p)) & \text{if } n = mp^{2k}, \ \ p \nmid m, \\ \frac{p-1}{p^{1+k}}\left(1 + \frac{1}{p}\right) & \text{if } n = mp^{2k+1}, \ \ p \nmid m. \end{cases} \tag{2.7}$$

3. Computing some local representation densities. The non-dyadic case.

In this section we prove

Theorem 3.1. *Let p be an odd prime and u be any integer with $(-u|p) = -1$. Let G be some ternary genus such that*

$$\tilde{f} \sim_p ux^2 + p(y^2 + uz^2)$$

for any $\tilde{f}$ in G. Then

$$d_{G,p}(n) = \frac{p}{p-1}\Gamma_p(n). \tag{3.1}$$

Here, and everywhere, the relation $\tilde{f} \sim_p \tilde{g}$ means that the two quadratic forms $\tilde{f}$ and $\tilde{g}$ are equivalent over the p-adic integers $\mathbf{Z}_p$.

It may not be obvious that the above theorem is identical to the special case $\epsilon_p = -1$ of Lemma 4.2 in [5]. Here we take a leisurely and self-contained approach, counting solutions of the relevant equation modulo p^t for large t. Suppose

$$ux^2 + p(y^2 + uz^2) \equiv n \pmod{p^t} \text{ with } p^2 \mid n, \quad 2 \leq t.$$

Then, thanks to $(-u|p) = -1$, we have $p \mid x$, $p \mid y$, $p \mid z$. This observation implies that

$$d_{G,p}(np^{2k}) = \frac{d_{G,p}(n)}{p^k}, \text{ if } p^2 \nmid n. \tag{3.2}$$

Hence (3.1) holds true for all n if it holds true for all n such that $p^2 \nmid n$. There are two cases to consider. First, when $p \nmid n$ we have that

$$\begin{aligned}
&|\{(x,y,z) \in Z^3 : 0 \leq x,y,z < p^t, ux^2 + p(y^2+uz^2) \equiv n \pmod{p^t}\}| \\
=&|\{(x,y,z) \in Z^3 : 0 \leq x,y,z < p^t, x^2 \equiv un - up(y^2+uz^2) \pmod{p^t}\}| \\
=&\sum_{y=0}^{p^t-1}\sum_{z=0}^{p^t-1}(1 + ((un - up(y^2+uz^2))|p)) = p^{2t}(1 + (un|p)) = p^{2t}(1 - (-n|p)).
\end{aligned}$$

And so

$$d_{G,p} = \frac{1}{p^{2t}}p^{2t}(1 - (-n|p)) = (1 - (-n|p)). \tag{3.3}$$

We comment that in the discussion above we used a well known

Lemma 3.2. *Let p' be an odd prime not dividing c. The number of roots of*

$$x^2 \equiv c \pmod{p'^t}, \quad t \geq 1$$

is the same as the number $(0$ or $2)$ of roots when $t = 1$. That is

$$|\{0 \leq x < p^t : x^2 \equiv c \pmod{p^t}\}| = 1 + (c|p).$$

This lemma is proven in [8].
Second, when $n = pm$ and $p \nmid m$ we have that

$$
\begin{aligned}
&|\{(x,y,z) \in Z^3 : 0 \leq \ x,y,z < p^t, ux^2 + p(y^2+uz^2) \equiv pm \pmod{p^t}\}| \\
=&p^2 \ |\{(x,y,z) \in Z^3 : 0 \leq \ x,y,z < p^{t-1}, upx^2 + y^2 + uz^2 \equiv m \pmod{p^{t-1}}\}| \\
=&p^2 |\{(x,y,z) \in Z^3 : 0 \leq \ x,y,z < p^{t-1}, y^2 \equiv m - upx^2 - uz^2 \pmod{p^{t-1}}\}| \\
=&p^2 \sum_{x=0}^{p^{t-1}-1} \sum_{z=0}^{p^{t-1}-1} (1 + ((m - upx^2 - uz^2)|p)) \\
=&p^{2t} + p^2 \sum_{x=0}^{p^{t-1}-1} \sum_{z=0}^{p^{t-1}-1} ((m - upx^2 - uz^2)|p) \\
=&p^{2t} + p^{t+1} \sum_{z=0}^{p^{t-1}-1} ((m - uz^2)|p) = p^{2t} - p^{t+1} \sum_{z=0}^{p^{t-1}-1} ((-um + z^2)|p) \\
=&p^{2t} + p^{t+1} p^{t-2} = p^{2t}\big(1 + \frac{1}{p}\big).
\end{aligned}
$$

This time we used another well known fact:

$$
\sum_{y=0}^{p-1} ((y^2 + a)|p) = -1, \tag{3.4}
$$

with p being an odd prime not dividing a.
Hence

$$
d_{G,p} = \frac{1}{p^{2t}} p^{2t} \big(1 + \frac{1}{p}\big) = \big(1 + \frac{1}{p}\big), \tag{3.5}
$$

as desired.
Our proof of Theorem 3.1 is now complete.

4. Computing some local representation densities. The dyadic case.

In this section we prove two theorems.

Theorem 4.1. *Let G_1 be some ternary genus such that*

$$\tilde{f} \sim_2 yz - x^2$$

for any $\tilde{f}$ in G_1. Let $n = 4^a k, 4 \nmid k$, then

$$
d_{G_1,2}(n) = \begin{cases} \frac{3}{2} & \text{if } k \equiv 7 \pmod 8, \\ \frac{3}{2} - \frac{1}{2^{a+1}} & \text{if } k \equiv 3 \pmod 8, \\ \frac{3}{2} - \frac{3}{2^{a+2}} & \text{if } k \equiv 1, 2 \pmod 4. \end{cases} \tag{4.1}
$$

Theorem 4.2. *Let G_2 be some ternary genus such that*

$$\tilde{f} \sim_2 4yz - x^2$$

for any $\tilde{f}$ in G_2. Let $n = 4^a k, 4 \nmid k$, then

$$d_{G_2,2}(n) = \begin{cases} 3 & \textit{if } k \equiv 7 \pmod 8, \\ 3 - \frac{1}{2^{a-1}} & \textit{if } k \equiv 3 \pmod 8, \\ 3 - \frac{3}{2^a} & \textit{if } k \equiv 1, 2 \pmod 4. \end{cases} \tag{4.2}$$

Comparing (1.5), (4.1), and (4.2) we have at once

$$\psi(n) = 2d_{G_1,2}(n) - d_{G_2,2}(n). \tag{4.3}$$

We note two related recurrences

$$2d_{G_1,2}(n) - d_{G_2,2}(4n) = 0 \tag{4.4}$$

and

$$4d_{G_1,2}(n) - d_{G_2,2}(n) = 3. \tag{4.5}$$

To prove Theorem 4.1 and Theorem 4.2, it is sufficient to show that (4.4) and (4.5), together with the initial conditions

$$d_{G_2,2}(n) = \begin{cases} 3 & \text{if } n \equiv 7 \pmod 8, \\ 1 & \text{if } n \equiv 3 \pmod 8, \\ 0 & \text{if } n \equiv 1, 2 \pmod 4, \end{cases} \tag{4.6}$$

hold true. Note that (4.4) follows easily from

$$\begin{aligned} &|\{(x,y,z) \in Z^3 : 0 \le x,y,z < 2^t, 4yz - x^2 \equiv 4n \pmod{2^t}\}| \\ =&2 \cdot 4 \cdot 4 |\{(x,y,z) \in Z^3 : 0 \le x,y,z < 2^{t-2}, yz - x^2 \equiv n \pmod{2^{t-2}}\}|. \end{aligned}$$

Clearly, when $n \equiv 1, 2 \pmod 4$ we have

$$4yz - n \equiv 2, 3, 6, 7 \pmod 8.$$

Recalling Lemma 2.1, we see that the congruence

$$4yz - x^2 \equiv n \pmod{2^t}$$

has no solutions when $t \ge 3$. Consequently,

$$d_{G_2,2}(n) = 0 \quad \text{if } n \equiv 1, 2 \pmod 4.$$

Next, when $n \equiv 3 \pmod 8$ we have

$$\begin{aligned} &|\{(x,y,z) \in Z^3 : 0 \le x,y,z < 2^t, 4yz - x^2 \equiv n \pmod{2^t}\}| \\ =&4|\{(y,z) \in Z^2 : 0 \le y,z < 2^t, yz \equiv 1 \pmod 2\}| = 4 \cdot 2^{t-1}2^{t-1} = 2^{2t}. \end{aligned}$$

Hence

$$d_{G_2,2}(n) = 1 \ \text{ if } n \equiv 3 \pmod 8.$$

The case $n \equiv 7 \pmod 8$ in (4.6) can be treated in an analogous manner.

$$\begin{aligned} &|\{(x,y,z) \in Z^3 : 0 \le x,y,z < 2^t, 4yz - x^2 \equiv n \pmod{2^t}\}| \\ =&4\ |\{(y,z) \in Z^2 : 0 \le y,z < 2^t, yz \equiv 0 \pmod 2\}| = 4 \cdot (2^t 2^t - 2^{t-1}2^{t-1}) = 3 \cdot 2^{2t}. \end{aligned}$$

Hence

$$d_{G_2,2}(n) = 3 \text{ if } n \equiv 7 \pmod 8.$$

And so we have established the initial conditions (4.6). It remains to prove (4.5). We shall require the following easy companion to Lemma 2.1

Lemma 4.3. *Let $t = 2s + 3 + \delta$, with integers $0 \le \delta \le 1$ and $0 \le s$.
Let $S_t(c) := |\left\{0 \le x < 2^t : \ x^2 \equiv c \pmod{2^t}\right\}|$.
Then*

$$S_t(4^m \cdot (8r+1)) = \begin{cases} 2^{m+2} & \text{if } \ 0 \le m \le s, \\ 2^{s+1+\delta} & \text{if } \ m = s+1, \\ 2^{s+1+\delta} & \text{if } \ m = s+2. \end{cases}$$

When $m = s + 2$ the formula $4^m \cdot (8r + 1)$ refers to 0, as then $2s + 4 \ge t$ and $4^m \ge 2^t$; in fact $2^t \mid 4^m$. It is important to note that no $0 \le c < 2^t$ other than the specified values above are allowed to have $S_t(c) \ne 0$.

To proceed further we define

$$P_{i,t}(c) = |\left\{(y,z) \in Z^2 : 0 \le y, z < 2^t, 4^{i-1}yz \equiv c \pmod{2^t}\right\}|, \quad i = 1, 2,$$

and

$$C_{m,t} = \left\{c \in Z : 0 \le c < 2^t, c \equiv 4^m \pmod{8 \cdot 4^m}\right\}, \quad m \in Z.$$

Again we comment that when $m = s+2$ the condition $c \equiv 4^m \pmod{8 \cdot 4^m}$ means $c = 0$. It is not hard to check that

$$4 \cdot P_{1,t}(n) - P_{2,t}(n) = \begin{cases} 2^{t+1} & \text{if } n \equiv 1 \pmod 2, \\ 2^{t+2} & \text{if } n \equiv 0 \pmod 2, \end{cases} \tag{4.7}$$

and that for $t = 2s + 3 + \delta$, with integers $0 \le \delta \le 1$ and $0 \le s$

$$|C_{m,t}| = \begin{cases} 2^{2s-2m+\delta} & \text{if } 0 \le m \le s, \\ 1 & \text{if } m = s+1, \\ 1 & \text{if } m = s+2. \end{cases} \tag{4.8}$$

Next, we define for $i = 1, 2$

$$L_{i,t}(c) = |\{(x,y,z) \in Z^3 : 0 \le x, y, z < 2^t, \quad 4^{i-1}yz - x^2 \equiv c \pmod{2^t}\}|.$$

From Lemma 4.3 it is easy to see that

$$L_{i,t}(n) = \sum_{m=0}^{s+2} \sum_{c \in C_{m,t}} S(c) P_{i,t}(n+c), \quad i = 1, 2.$$

Making use of Lemma 4.3, (4.7), and (4.8) we find that

$$4 \cdot L_{1,t}(n) - L_{2,t}(n) = \sum_{m=0}^{s+2} \sum_{c \in C_{m,t}} S(c) \cdot (4 \cdot P_{1,t}(n+c) - P_{2,t}(n+c)) = 3 \cdot 4^t. \tag{4.9}$$

Finally, we note that for sufficiently large t

$$d_{G_i,2}(n) = \frac{1}{4^t} L_{i,t}(n), \quad i = 1, 2. \tag{4.10}$$

Combining (4.9) and (4.10) we see that (4.5) holds true. Our proofs of Theorem 4.1 and Theorem 4.2 are now complete.

5. Computing the mass of the ternary genus $TG_{1,p}$

In this section we prove that

$$M(TG_{1,p}) = \frac{p-1}{48}, \tag{5.1}$$

where p is a fixed odd prime and

$$M(TG_{1,p}) := \sum_{\tilde{f}\in TG_{1,p}} \frac{1}{|\mathrm{Aut}(\tilde{f})|}.$$

To prove (5.1) we will employ the Smith-Minkowski-Siegel mass formula. This formula gives the mass as an infinite product over all primes. Many published versions of this formula have small errors. In this paper we will follow a reliable account by Conway and Sloane [7]. From equation (2) in [7] we have that

$$M(TG_{1,p}) = \frac{1}{\pi^2}\prod_{p'} 2m_{p'}, \tag{5.2}$$

where p' runs through all primes and where local masses m_p' are defined in equation (3) in [7] by

$$m_{p'} = \prod_{q} M_q \prod_{q<q'} (q'/q)^{\frac{n(q)n(q')}{2}} 2^{n(I,I)-n(II)}. \tag{5.3}$$

Here q ranges over all powers p'^t of p' (including those with negative t). The last factor in (5.3) is 1 for all odd primes. So if the p'-adic Jordan decomposition of $\tilde{f}\in TG_{1,p}$ is given by

$$\sum_q q\tilde{f}_q,$$

then

$$n(q) = \dim(\tilde{f}_q).$$

For all odd primes p' such that $p'\nmid p$, the p'-adic Jordan decomposition of any form $\tilde{f}\in TG_{1,p}$ can be taken to be $(x^2+y^2+z^2)$; this follows from Theorem 29 in [13]. So with the aid of Table 2 in [7] we find that

$$n(1) = 3, \quad M_1 = \frac{p'^2}{2(p'^2-1)}.$$

If $q\neq 1$ then

$$n(q) = 0, \quad M_q = 1.$$

Hence

$$m_{p'} = M_1 = \frac{p'^2}{2(p'^2-1)}, \quad p'\nmid 2p. \tag{5.4}$$

Next, the p-adic Jordan decomposition of any form $\tilde{f}\in TG_{1,p}$ is given by $\tilde{f}_1 + p\tilde{f}_p$ with $\tilde{f}_1 = ux^2$, $\tilde{f}_p = (y^2+uz^2)$, for a unit u satisfying $(-u|p) = -1$ (see Section 6). And so from Table 1 and Table 2 in [7], we see that $n(1) = 1$, species$(1) = 1$, $M_1 = \frac{1}{2}$, and $n(p) = 2$, species$(p) = 2$, $M_p = \frac{p}{2(p+1)}$. And so we find that

$$m_p = M_1 M_p (p/1)^{\frac{2}{2}} = \frac{p^2}{4(p+1)}. \tag{5.5}$$

Finally, one possible 2-adic Jordan decomposition of any form $\tilde{f} \in TG_{1,p}$ is given by

$$\frac{1}{2}\tilde{f}_{\frac{1}{2}} + \tilde{f}_1 + 2\tilde{f}_2,$$

with $\tilde{f}_{\frac{1}{2}} = 2yz$, $\tilde{f}_1 = -x^2$, $\tilde{f}_2 = 0$. This follows from Theorem 29 in [13]. We note that $\tilde{f}_2$ is a bound love form. It contributes a factor of $\frac{1}{2}$ to the mass

$$M_2 = \frac{1}{2}.$$

Obviously $n\big(\frac{1}{2}\big) = 2$, $n(1) = 1$, and $n(2) = 0$.
Next, $\tilde{f}_{\frac{1}{2}}$ is of the type II_2. It is bound and has octane value $= 0$, species $= 3$, $M_{\frac{1}{2}} = \frac{2}{3}$. Also, $\tilde{f}_1$ is of the type I_1. It is free and has octane value $= 0 - 1 = -1$, species $= 0+$, $M_1 = 1$. In (5.3), $n(\mathrm{I},\mathrm{I})$ is the total number of pairs of adjacent constituents $\tilde{f}_q$, $\tilde{f}_{2q}$ that are both of type I and $n(\mathrm{II})$ is the sum of the dimensions of all Jordan constituents that have type II. Clearly $n(\mathrm{I},\mathrm{I}) = 0$ and $n(\mathrm{II}) = 2$. So

$$m_2 = \frac{2}{3}(1)\frac{1}{2}(2/1)^{\frac{2}{2}}2^{0-2} = \frac{1}{6}. \tag{5.6}$$

Combining (5.2)–(5.6), we obtain

$$M(TG_{1,p}) = \frac{1}{\pi^2}\frac{1}{3}\frac{p^2}{2(p+1)}\prod_{\gcd(2p,p')=1}\frac{p'^2}{p'^2-1} = \frac{p-1}{8\pi^2}\prod_{p'}\frac{p'^2}{p'^2-1}.$$

Recalling that

$$\prod_{p'}\frac{p'^2}{p'^2-1} = \sum_{n\geq 1}\frac{1}{n^2} = \frac{\pi^2}{6},$$

we see that (5.1) holds true.

6. A Tale of Two Genera

Here we will give an overview of the construction of $TG_{2,p}$. We are given $TG_{1,p}$. The sextuple

$$\langle a, b, c, d, e, f\rangle$$

refers to

$$ax^2 + by^2 + cz^2 + dyz + ezx + fxy,$$

with Gram matrix

$$\begin{pmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{pmatrix}. \tag{6.1}$$

First we will show that any form in $TG_{1,p}$ is equivalent to a form in Convenient Shape 1, which is just $\langle a, b, c, d, e, f\rangle$ with $a \equiv 3 \pmod 4$, then d odd and e, f even. Any primitive form represents an odd number, therefore it primitively represents an odd number a, so we may insist that a be odd. A particularly simple operation taking a form to an equivalent one is constructing the form with Gram matrix $M_{ij}'GM_{ij}$, where G is the current Gram matrix of the form and M_{ij} is the result of

beginning with the identity matrix and placing a single 1 at position ij, and M'_{ij} denotes the transpose of M_{ij}. We can also permute variables with the matrix

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.2}$$

If e and f are both even we are done. Otherwise, at least one of them is odd. If f is the odd one apply M_0, as in $M'_0 G M_0$, to arrive at $\langle a, c, b, -d, -f, e\rangle$; call these the new values of all the letters a, b, c, d, e, f. If d is even, apply M_{12} to get odd d in $\langle a, a+b+f, c, d+e, e, f+2a\rangle$. If f is odd apply M_{32} to get even f in $\langle a, b+c+d, c, d+2c, e, f+e\rangle$. From the definition of the discriminant, $\Delta = 4abc + def - ad^2 - be^2 - cf^2$, with $\Delta = p^2 \equiv 1 \pmod 4$; it follows that b is even, so we have b, f even and a, d odd. Finally, apply M_{21} to get even e in $\langle a+b+f, b, c, d, e+d, f+2b\rangle$, where the new value of a is still odd, while e has become even. With a, d odd and e, f even, all the terms in $\Delta = 4abc + def - ad^2 - be^2 - cf^2$ are divisible by 4 except $-ad^2$. Since $d^2 \equiv 1 \pmod 4$ and $\Delta \equiv 1 \pmod 4$, it follows that $a \equiv 3 \pmod 4$. Given a primitive form $\langle a, b, c, d, e, f\rangle$ in Convenient Shape 1, define a mapping Φ giving another primitive form by

$$\Phi\left(\langle a, b, c, d, e, f\rangle\right) = \langle a, 4b, 4c, 4d, 2e, 2f\rangle.$$

Note that if $g(x, y, z)$ is in Convenient Shape 1 and $h = \Phi(g)$, then $h(x, y, z) = g(x, 2y, 2z)$. Any primitive form $\langle a, b, c, d, e, f\rangle$ with $d, e, f \equiv 0 \pmod 2$ that does not represent any number $n \equiv 1, 2 \pmod 4$ has $\Delta \equiv 0 \pmod{16}$ can be put in (is equivalent to a form in) Convenient Shape 2, that is with b, c, d, e, f all divisible by 4, and with $a \equiv 3 \pmod 4$.

To save space, we will introduce matrices

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $DE = ED = 2I$. At this point we will give an outline of the proof that Φ is a well-defined bijection between $TG_{1,p}$ and $TG_{2,p}$, that Φ preserves automorphs, that $TG_{2,p}$ is in fact a genus, and finally give the p-adic diagonalization for all these forms, along with a 2-adic Jordan decomposition. Given a primitive form g in Convenient Shape 1, with Gram matrix G, the Gram matrix for $\Phi(g)$ is EGE. Furthermore $\Phi(g)$ is also primitive. Now, suppose that forms g, h in $TG_{1,p}$ are equivalent and are already in Convenient Shape 1. Suppose they have Gram matrices G and H,respectively. So we are saying there is an integral matrix P such that $P'GP = H$. It turns out that matrix elements p_{21}, p_{31} are even, and so $\frac{1}{2}DPE$ is integral. But then

$$\begin{aligned} \left(\frac{1}{2}DPE\right)'\left(EGE\right)\left(\frac{1}{2}DPE\right) &= \left(\frac{1}{2}EP'D\right)\left(EGE\right)\left(\frac{1}{2}DPE\right) \\ &= EP'GPE \\ &= EHE. \end{aligned} \tag{6.3}$$

That is, the equivalence class of $\Phi(g)$ does not depend on the particular choice of Convenient Shape 1. Therefore Φ extends to a well defined mapping from the equivalence classes of forms in $TG_{1,p}$ to forms with $\Delta = 16p^2$ that are classically integral and do not represent any numbers n with $n \equiv 1, 2 \pmod 4$.
Now, let us take the collection of all the forms with $\Delta = 16p^2$ that are primitive, classically integral, and do not represent any numbers n with $n \equiv 1, 2 \pmod 4$ and call that $TG_{2,p}$. It is not difficult to show that such forms can be put into Convenient Shape 2, with Gram matrix H. Then $\frac{1}{4}DHD$ is the Gram matrix of a form in $TG_{1,p}$. It is not difficult to show that this "downwards" map also respects equivalence classes of forms, and the choice of Convenient Shape 2 does not matter. Therefore it is legitimate to name this mapping Φ^{-1}. As Φ and Φ^{-1} really are inverses, it follows that both are injective and surjective. Suppose g and h are in $TG_{1,p}$ and in Convenient Shape 1, with Gram matrices G_1 and H_1, repectively. As they are in the same genus, there is an odd number w not divisible by p, along with an integral matrix R, such that

$$R'G_1R = w^2 H_1,$$

and

$$\det G_1 = \det H_1.$$

This is Siegel's definition of a genus: rational equivalence "without essential denominator". Let $\Phi(g)$ have Gram matrix G_2, while $\Phi(h)$ has Gram matrix H_2. Then $Q = \frac{1}{2}DRE$ is integral, and we have

$$Q'G_2Q = w^2 Q_1.$$

That is, $\Phi(g)$ and $\Phi(h)$ are in the same genus, which we are calling $TG_{2,p}$. Next, if A is an automorph of $g \in TG_{1,p}$, in Convenient Shape 1, with Gram matrix G, this means that A has determinant ± 1 and

$$A'GA = G.$$

So, repeating (6.3), we find that $B = \frac{1}{2}DAE$ is an automorph of $\Phi(g)$. At the same time, beginning with $h \in TG_{2,p}$, in Convenient Shape 2, with Gram matrix H, and an automorph B solving $B'HB = H$, then $A = \frac{1}{2}EQD$ is an automorph of $\Phi^{-1}(h)$. That is, the number of automorphs are the same, from which it follows that the mass of $TG_{2,p}$ is exactly the same as the mass of $TG_{1,p}$. That is,

$$M(TG_{2,p}) = M(TG_{1,p}) = \frac{p-1}{48}. \tag{6.4}$$

A very similar formalism shows directly that

$$R_g(n) = R_{\Phi(g)}(4n),$$

where these are the (finite) number of representations by the indicated form. Now, from Theorem 29 in [13], we know that all forms in $TG_{1,p}$ are equivalent over the 2-adic integers to $yz - x^2$, or $\langle -1, 0, 0, 1, 0, 0 \rangle$ which is integral and is in Convenient Shape 1. The same process that took some $g \in TG_{1,p}$ and constructed automorphs or equivalences involving $\Phi(g)$ can be readily extended to the 2-adic integers. So we begin with $g \sim_2 \langle -1, 0, 0, 1, 0, 0 \rangle$ which shows that

$$\Phi(g) \sim_2 \Phi(\langle -1, 0, 0, 1, 0, 0 \rangle) = \langle -1, 0, 0, 4, 0, 0 \rangle.$$

So it follows that for any $h \in TG_{2,p}$, h is equivalent over the 2-adic integers to $4yz-x^2$, which is in Convenient Shape 2. So we have proved the following identities

$$\begin{aligned} &g \sim_2 yz - x^2, \;\; g \in TG_{1,p}, \\ &h \sim_2 4yz - x^2, \;\; h \in TG_{2,p}. \end{aligned} \tag{6.5}$$

Now to the p-adic diagonalization of these forms, which requires more terminology. The forms in either genus are isotropic in the 2-adic field, as there are nontrivial integral expressions with $yz - x^2 = 0$ or $4yz - x^2 = 0$. It follows that the forms in both genera are anisotropic (not "zero forms") in the p-adic field. This is from Lemma 1.1 in [6], page 76. What sort of numbers are represented by these forms? According to Corollary 13 in [9], page 41, some number n is represented by $g(x, y, z) \in TG_{1,p}$ in $\mathbf{Q}_p$ if and only if

$$h(x, y, z, w) = g(x, y, z) - nw^2$$

is isotropic in $\mathbf{Q}_p$. The determinant of h is $-np^2$. This is a square in $\mathbf{Q}_p$ if $(-n|p) = 1$. We already know that $c_p(h) = c_p(g) = -1$ by Lemma 2.3(iii) in [6], page 58. Thus $c_p(h) = -(-1, -1)_p$. Here $(a, b)_p$ is the Hilbert Norm Residue Symbol. By Lemma 2.6 on page 59, when $(-n|p) = 1$ we have $h(x, y, z, w) = g(x, y, z) - nw^2$ anisotropic in $\mathbf{Q}_p$ and so n is not represented. So, for $p \equiv 1 \pmod 4$, forms in $TG_{1,p}$ and $TG_{2,p}$ represent only quadratic nonresidues modulo p, among the numbers not divisible by p. For $p \equiv 3 \pmod 4$, forms in $TG_{1,p}$ and $TG_{2,p}$ represent only quadratic residues modulo p. Let p be an odd prime and u be any integer with $(-u|p) = -1$. From the fact that any binary form $ax^2 + bxy + cy^2$ with discriminant not divisible by p represents both residues and nonresidues modulo p, it follows that $g \in TG_{1,p}$ diagonalizes as $ux^2 + p(vy^2 + wz^2)$. Now, given any number V with $(-V|p) = 1$, it follows that $x_1^2 + V x_2^2$ is isotropic in $\mathbf{Q}_p$. As our forms are anisotropic there, it follows that $(-vw|p) = -1$. Meanwhile, as v, w are units in $\mathbf{Q}_p$, we know that the binary $vy^2 + wz^2$ represents both residues and nonresidues modulo p. From Lemma 3.4 in [6], page 115, we can insist that $v = 1$ and $w = u$, so that

$$g \sim_p ux^2 + p(y^2 + uz^2), \;\; \text{with } (-u|p) = -1, \;\; g \in TG_{1,p}. \tag{6.6}$$

By the usual methods, the same applies to $TG_{2,p}$. And so

$$h \sim_p ux^2 + p(y^2 + uz^2), \;\; \text{with } (-u|p) = -1, \;\; h \in TG_{2,p}. \tag{6.7}$$

It turns out that our bijection Φ is an instance of a Watson transformation [14]. As a result, the bijection generalizes to positive ternary forms with any odd discriminant. The phenomenon of bijection of automorphs and equal mass then generalizes to any dimension and any discriminant. We will discuss this further in Section 8.

7. Proof of Theorem 1.3.

Here we prove our main result (1.3). We recall that, thanks to Theorem 29 in [13], we have

$$\tilde{f} \sim_{p'} x^2 + y^2 + z^2,$$

for any ternary form $\tilde{f}$ with discriminant Δ, provided prime $p' \nmid 2\Delta$.
Hence

$$d_{TG_{1,p},p'} = d_{TG_{2,p},p'} = d_{x^2+y^2+z^2,p'}, \;\; p' \nmid 2p. \tag{7.1}$$

Next, we employ (2.1), (2.6), (3.1), (4.3), (6.4), (6.5), (6.6), (6.7), and (7.1) to rewrite the expression on the right of (1.3) as

$$\begin{aligned}
&\mathrm{RHS}(1.3) = \\
&\frac{p-1}{48} 96\pi \sqrt{\frac{n}{p^2}} (2d_{TG_{1,p},2}(n) - d_{TG_{2,p},2}(n)) \frac{p}{p-1} \Gamma_p(n) \prod_{\gcd(p',2p)=1} d_{x^2+y^2+z^2,p'}(n) \\
&= 2\pi\sqrt{n}(2d_{TG_{1,p},2}(n) - d_{TG_{2,p},2}(n))\Gamma_p(n) \prod_{\gcd(p',2p)=1} d_{x^2+y^2+z^2,p'}(n) \\
&= 2\pi\sqrt{n}\psi(n)\Gamma_p(n) \prod_{\gcd(p',2p)=1} d_{x^2+y^2+z^2,p'}(n) = \mathrm{LHS}(1.3).
\end{aligned}$$

This completes our proof of the Theorem 1.3.
We conlude with the following example. Genus $TG_{1,73}$ consists of four classes

$$TG_{1,73} = \{\mathrm{Cl}(h_1), \mathrm{Cl}(h_2), \mathrm{Cl}(h_3), \mathrm{Cl}(h_4)\},$$

where

$$\begin{aligned}
h_1(x,y,z) &= 31x^2 + 5y^2 + 11z^2 + yz - 14zx + 6xy, |\mathrm{Aut}(h_1)| = 2, \\
h_2(x,y,z) &= 15x^2 + 14y^2 + 10z^2 + 7yz + 4zx + 16xy, |\mathrm{Aut}(h_2)| = 2, \\
h_3(x,y,z) &= 11x^2 + 7y^2 + 20z^2 + 7yz + 2zx + 4xy, |\mathrm{Aut}(h_3)| = 4, \\
h_4(x,y,z) &= 7x^2 + 11y^2 + 21z^2 + 11yz + 2zx + 4xy, |\mathrm{Aut}(h_4)| = 4.
\end{aligned}$$

Note that all four forms above are in Convenient Shape 1. And so, we can immediately construct the second genus

$$TG_{2,73} = \{\mathrm{Cl}(g_1), \mathrm{Cl}(g_2), \mathrm{Cl}(g_3), \mathrm{Cl}(g_4)\},$$

where

$$\begin{aligned}
g_1(x,y,z) &= 31x^2 + 20y^2 + 44z^2 + 4yz - 28zx + 12xy, |\mathrm{Aut}(g_1)| = 2, \\
g_2(x,y,z) &= 15x^2 + 56y^2 + 40z^2 + 28yz + 8zx + 32xy, |\mathrm{Aut}(q_2)| = 2, \\
g_3(x,y,z) &= 11x^2 + 28y^2 + 80z^2 + 28yz + 4zx + 8xy, |\mathrm{Aut}(g_3)| = 4, \\
g_4(x,y,z) &= 7x^2 + 44y^2 + 84z^2 + 44yz + 4zx + 8xy, |\mathrm{Aut}(h_4)| = 4.
\end{aligned}$$

From (1.3) we get

$$\begin{aligned}
s(73^2 n) - 73s(n) &= 24(31,5,11,1,-14,6)(n) + 24(15,14,10,7,4,16)(n) \\
&+ 12(11,7,20,7,2,4)(n) + 12(7,11,21,11,2,4)(n) \\
&- 48(31,20,44,4,-28,12)(n) - 48(15,56,40,28,8,32)(n) \\
&- 24(11,28,80,28,4,8)(n) - 24(7,44,84,44,4,8)(n).
\end{aligned} \tag{7.2}$$

8. Watson's Transformations

We begin with a brief summary of the Watson "m-mapping".

Let f be an n-ary quadratic form. In [14] equation (2.4), Watson defines a certain lattice $\Lambda_m(f)$ by $\vec{x} \in \Lambda_m(f)$ if and only if $f(\vec{x}+\vec{z}) \equiv f(\vec{z}) \pmod{m}$, $\forall \vec{z}$. This is a lattice, because if $\vec{x} \in \Lambda_m(f)$, then $-\vec{x} \in \Lambda_m(f)$, and if $\vec{x_1}, \vec{x_2} \in \Lambda_m(f)$, then $\vec{x_1}+\vec{x_2} \in \Lambda_m(f)$. Let F be the Gram (or second partials) matrix for the n-ary form f. This means, with column vector $\vec{x}$ and its transposed row vector $\vec{x}'$, that $f(\vec{x}) = \frac{1}{2}\,\vec{x}'F\vec{x}$. Watson also supplies his (2.2), which is simply $f(\vec{x}+\vec{z}) = f(\vec{x}) + \vec{z}'F\vec{x} + f(\vec{z})$. Now, we can vary $\vec{z}$ arbitrarily in $f(\vec{x}) + \vec{z}'F\vec{x} + f(\vec{z}) \equiv f(\vec{z}) \pmod{m}$. So $\vec{x} \in \Lambda_m(f)$ if and only if $f(\vec{x}) \equiv 0 \pmod{m}$ and $F\vec{x} \equiv \vec{0} \pmod{m}$, thus giving $n+1$

equations $\pmod{m}$, the first one tending to be redundant (depending on powers of 2 in m and various coefficients). Then Watson says to take a square integral matrix M whose columns serve as an integral basis for $\Lambda_m(f)$, and in (2.5) defines the result of the "m-mapping", that we denote $g = \lambda_m(f)$, by $g(\vec{y}) = \frac{1}{m}\, f(M\vec{y})$. So, if we take G as the Gram matrix for g, we have $G = \frac{1}{m} M'FM$, where M' refers to the transpose of M. By construction, G is an integral matrix, as all entries of FM are divisible by m, and $g(\vec{y}) = \frac{1}{2}\, \vec{y}'G\vec{y}$. Watson shows that a different choice of basis matrix M simply gives a form equivalent to g. So this is Watson's "m-mapping".

We are ready to show that our bijection Φ is Watson's λ_4. Let our discriminant $\Delta = \delta$ be odd, and let our form $\tilde{f}$ be in the analogue of convenient shape 1. That is, a, d odd and e, f even in $\langle a, b, c, d, e, f\rangle$. Notice that

$$\delta \equiv -ad^2 \equiv -a \pmod{4}.$$

For the case of $TG_{1,p}$ we had $\delta \equiv 1 \pmod{4}$ so it followed that, for $TG_{1,p}$, in convenient shape 1 we had $a \equiv 3 \pmod{4}$. But we will need only odd a. We just use $a \equiv -\delta \pmod{4}$. We find that $\vec{x} \in \Lambda_4(\tilde{f})$ when $\tilde{f}(\vec{x}) \equiv 0 \pmod{4}$ and

$$F\vec{x} \equiv \vec{0} \pmod{4}.$$

Here $\vec{x}$ is the column vector

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

So we have three linear equations $\pmod{4}$. From a, d odd, e, f even, and $fx + 2by + dz \equiv 0 \pmod{4}$ we find that z is even. From $ex + dy + 2cz \equiv 0 \pmod{4}$ we find that y is also even. With y, z even, a, d odd, e, f even, and $2ax + fy + ez \equiv 0 \pmod{4}$ we find that x is even. With x, y, z even, a, d odd, e, f even, back to $fx + 2by + dz \equiv 0 \pmod{4}$, we find that $z \equiv 0 \pmod{4}$. With x, y, z even, a, d odd, e, f even, back to $ex + dy + 2cz \equiv 0 \pmod{4}$, we find that $y \equiv 0 \pmod{4}$. Finally, with x, y, z all even, we do indeed have $\tilde{f}(\vec{x}) \equiv 0 \pmod{4}$.

Now, look at Watson's (2.5), his M must have columns that form an integral basis of $\Lambda_4(\tilde{f})$, and this particular time it makes sense to choose a matrix already in Smith Normal Form,

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \tag{8.1}$$

Back to Watson's (2.5), we get the result of the "m-mapping", that we write as $g = \lambda_4(\tilde{f})$, $g(\vec{y}) = \frac{1}{4}\, \tilde{f}(M\vec{y})$, or, reverting to vector (x, y, z)

$$g(x, y, z) = \frac{1}{4}\, \tilde{f}(2x, 4y, 4z) = \tilde{f}(x, 2y, 2z). \tag{8.2}$$

The coefficients for g are $\langle a, 4b, 4c, 4d, 2e, 2f\rangle$ so that all but the first are divisible by 4, and the form g cannot take any value $2, \delta \pmod{4}$, as $a \equiv -\delta \pmod{4}$. Meanwhile (8.2) shows that g represents a subset of the numbers represented by $\tilde{f}$. Also (8.2) shows that, in the case of $TG_{1,p}$, $\Phi = \lambda_4$.

Our new discriminant is $\Delta = 16\delta$, with any form g in convenient shape 2, that is $\langle a, b, c, d, e, f\rangle$ with some odd $a \equiv -\delta \pmod{4}$ and b, c, d, e, f divisible by 4, so that $g \neq 2, \delta \pmod{4}$. Note that, with odd δ, and $\Delta \equiv \pm 16 \pmod{64}$, we know that this $d \equiv 4 \pmod{8}$. So, if we call this form $g(x, y, z)$, we have $g(x, y, z) \equiv ax^2$

(mod 4). If we let $\vec{z}$ be the column vector with entries u, v, w, Watson's (2.4) for $\Lambda_4(g)$ becomes $a(x+u)^2 \equiv au^2 \pmod 4$, $\forall u$. So, $ax^2 + 2aux \equiv 0 \pmod 4$, $\forall u$, and it is necessary and sufficient to have the first coordinate x even. So this time, using the matrix name N, we get

$$N = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.3}$$

and $MN = NM = 4I$.

Let us take, say, $h = \lambda_4(g)$. Back to Watson's (2.5), we get $h(\vec{y}) = \frac{1}{4}\, g(N\vec{y})$, or, reverting to vector (x, y, z)

$$h(x, y, z) = \frac{1}{4}\, g(2x, y, z) = g(x, \frac{y}{2}, \frac{z}{2}).$$

So, once again, in the case of $TG_{1,p}$, $\Phi^{-1} = \lambda_4$. Furthermore, beginning with a genus of any odd discriminant δ, $\lambda_4^2(\tilde{f}) = \tilde{f}$.

We now proceed to describe a very general result that could easily have been present in [14] but is not there in any explicit manner, and indeed for which we have no reference. So this result about automorphs and mass may be new to some extent. Note that Watson uses (f) to refer to the integral equivalence class of the form f.

Theorem 8.1. *In any dimension n, with any m and any discriminant, if λ_m^2 is the identity on a class (f), then λ_m is a bijection between the sets of equivalence classes in the genus of f and the genus of $\lambda_m(f)$. Furthermore, λ_m induces a bijection of integral automorphs between any form f_1 in the genus of f and its image $\lambda_m(f_1)$. As a result, taking positive forms, the two genera have the same mass.*

Our proof is long but elementary, and can be written entirely using Watson's terminology and concepts. We content ourselves with a sketch of the proof. Let the columns of the matrix M be an integral basis for $\Lambda_m(f)$, and let $g = \lambda_m(f)$, so that if G as the Gram matrix for g, we have $G = \frac{1}{m}M'FM$. Watson shows that equivalent forms are mapped to equivalent forms, forms in the same genus are mapped to forms in the same genus, and that the map from genus to genus is surjective. It is not usually injective, and usually $\lambda_m^2\,((f))$ has a different discriminant from that of f, therefore usually can not even be in the same genus as f. If it should happen that $\lambda_m^2\,((f)) = (f)$, there is, in fact, a matrix N, where N turns out to be integral as well (Watson's Lemma 2(ii), page 581), such that

$$MN = NM = mI, \tag{8.4}$$

$$G = \frac{1}{m}M'FM, \tag{8.5}$$

$$F = \frac{1}{m}N'GN. \tag{8.6}$$

These formulae extend to a bijection between genera, where the actual matrix M depends on f, and of course N depends on M. Finally, if $\lambda_m(f) = g$, and if R is an integral automorph of f, that is $R'FR = F$, then

$$S = \frac{1}{m}NRM \tag{8.7}$$

is an integral automorph of g, that is $S'GS = G$. And if S is an integral automorph of g, then

$$R = \frac{1}{m} MSN \tag{8.8}$$

is an integral automorph of f. The point that required extra care in the proof was the fact that, in our situation, the columns of the matrix N really do give an integral basis for $\Lambda_m(g)$. The key to this is Watson's sentence on page 585, "Now the foregoing argument shows that we have $|H| = \pm 1$ and $f \sim \phi$ if and only if the lattice $\mu_m(f)$ is $m\Lambda_n$".This refers to his Theorem 2 on page 580.

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Department of Mathematics, University of Florida, Gainesville, Florida 32611-8105
E-mail address: alexb@ufl.edu

Math.Sci.Res.Inst., 17 Gauss Way, Berkeley, CA 94720-5070
E-mail address: jagy@msri.org